MASS ACADEMY

Worcester County Mathematics League

Varsity Meet 2
November 20, 2013

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Varsity Meet 2 – November 20, 2013
Round 1: Fractions, Decimals, and Percents

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Increasing 350 by a certain percentage gives the same number as decreasing 650 by the same percentage. What is the percentage?

2. Simplify:

$$\frac{\frac{4}{9} - 0.0\overline{4} + \frac{1}{5}}{3} \div \frac{2}{7}$$

(The notation $0.0\overline{4}$ indicates a repeating decimal.)

3. A delivery truck is full to start the day. At stop 1, 1/3 of the packages are delivered. At stop 2, 25% of the remaining packages are delivered. At stop 3, 50% of the remaining packages are delivered. If 6 more packages are delivered at stop 3 than at stop 2, how many packages are left on the truck after the delivery at stop 3?

ANSWERS

(1 pt.) 1. ______________________ %

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________ packages

Varsity Meet 2 – November 20, 2013
Round 2: Algebra I

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $5^{18} \times 2^{20}$ is multiplied out, what is the leading digit of the product?

2. If X and Y are whole numbers with $20 \le X \le 62$ and $39 \le Y \le 80$, what is the maximum possible value of

$$\frac{X+Y}{X}\ ?$$

3. Solve for all possible values of x:

$$\sqrt{4-\sqrt{x+7}} = \sqrt{9-x}$$

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

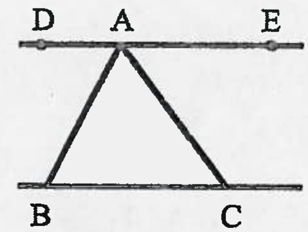
(3 pts.) 3. ____________________

Varsity Meet 2 – November 20, 2013
Round 3: Parallel Lines and Polygons

All answers must be in simplest exact form in the answer section

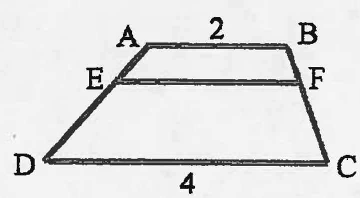
NO CALCULATOR ALLOWED

1. In the figure, if $\overline{DE} \parallel \overline{BC}$, $m\angle EAC = 40°$, and $m\angle ABC = 60°$, what is the degree measure of $\angle CAB$?



2. A quadrilateral is formed by connecting in order the midpoints of the sides of a rhombus whose diagonals are 20 and 12. How long is the longest diagonal of this quadrilateral?

3. Trapezoid $ABCD$, with base lengths $AB = 2$ and $CD = 4$, has area 3, and $\overline{EF} \parallel \overline{AB}$. If the areas of trapezoids $ABFE$ and $EFCD$ are in the ratio of 7 : 20, find the distance between $\overline{AB}$ and $\overline{EF}$.



ANSWERS

(1 pt.) 1. ____________________ °

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 2 – November 20, 2013
Round 4: Sequences and Series

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Find the 7^{th} term of the geometric sequence 96, -48, 24, ... (Treat 96 as the 1^{st} term, not the 0^{th} term.)

2. Consider the sequence $S = x + 2,\ 4x - 2,\ 6x - 3$, consisting of 3 non-zero terms. There exist two real numbers A and G such that S is an arithmetic sequence if $x = A$, and S is a geometric sequence if $x = G$. Find the value of $A + G$.

3. Evaluate the sum: $1(99) + 3(97) + 5(95) + 7(93) + \ldots + 99(1)$.

ANSWERS

(1 pt.) 1. ________________

(2 pts.) 2. ________________

(3 pts.) 3. ________________

Varsity Meet 2 – November 20, 2013
Round 5: Matrices and Systems of Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If **A** is a 5×7 matrix and **C** is a 5×8 matrix, find the dimensions of **B** if $\mathbf{A} \times \mathbf{B} = \mathbf{C}$. Specify the dimensions as #rows × #columns.

2. Find the product of all possible values of x which satisfy the system

$$\begin{cases} x^2 + y^2 + 6y = -5 \\ x^2 - y = 7 \end{cases}$$

3. Calculate $(A^\top)^{-1} + (A^{-1})^\top + (B + B^{-1})^\top$ given that

$$A = \begin{bmatrix} 5 & 8 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}.$$

($X^\top$ denotes the transpose of X.)

ANSWERS

(1 pt.) 1. ________ × ________ [#rows × #columns]

(2 pts.) 2. ____________________

(3 pts.) 3. $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

Varsity Meet 2 – November 20, 2013
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. The tens digit in a two-digit number is one less than the units digit. The product of the number and the number with its digits reversed is 736. Find the original number.

2. The admission fee to a math exhibit was \$15. When the fee was reduced, the number of customers per day increased by 50%. The amount of money collected per day increased by 25%. What was the reduced fee?

3. How many positive integers less than 1000 are not divisible by 5 or 7?

4. Four duck hunters, who are all perfect shots, go hunting when 4 ducks fly overhead. Each hunter randomly chooses a duck and they shoot simultaneously. What is the expected number of ducks that survive? (*i.e.* If this situation were repeated many times, what is the average number of ducks that are not targeted by any of the hunters?)

5. If half of the number which exceeds x by 3 is less than x by 25%, find x.

6. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$.

7. Algernon, Jack, and Ernest "solved" the same quadratic equation in x. The coefficient of x^2 was 1. Algernon copied the constant term incorrectly and got an answer of 7 and -3. Jack read the coefficient of x incorrectly and got 4 and -3. Ernest made no error and got the correct answer. What were the roots?

8. If the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 12$, find the value of $\begin{vmatrix} 2a_1 & a_3 & a_2 - 2a_3 \\ 2b_1 & b_3 & b_2 - 2b_3 \\ 2c_1 & c_3 & c_2 - 2c_3 \end{vmatrix}$.

9. Let P be the point inside regular pentagon $ABCDE$ such that $\triangle APE$ is equilateral. Find the degree measure of $\angle BPD$.

Varsity Meet 2 – November 20, 2013
TEAM ROUND ANSWER SHEET

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

1. ____________________

2. $ ____________________

3. ____________________

4. ____________________ ducks

5. ____________________

6. ____________________

7. ____________________

8. ____________________

9. ____________________ °

Varsity Meet 2 – November 20, 2013
ANSWERS

ROUND 1
(St. John's, QSC, Shrewsbury)

1. 30 or 30%
2. $0.7 = 7/10$
3. 18

ROUND 2
([unknown school], Auburn, North)

1. 4
2. 5
3. 9

ROUND 3
(Algonquin, Auburn, Doherty)

1. 80 or 80°
2. $2\sqrt{34}$
3. $1/3 = 0.\overline{3}$

ROUND 4
(Southbridge, QSC, Worc Acad)

1. $3/2 = 1.5 = 1\frac{1}{2}$
2. 5
3. 83350

ROUND 5
(Tahanto, St. Peter-Marian, Assabet Valley)

1. 7×8 (NOT 56)
2. 12
3. $\begin{bmatrix} 4 & 4 \\ 16 & 0 \end{bmatrix}$

TEAM ROUND
(Hudson, Shepherd Hill, Leicester, QSC, Bancroft, Leicester, Westborough, St. John's, Burncoat)

1. 23
2. \$12.50
3. 686
4. $81/64 = 1\frac{17}{64} \approx 1.266$
5. 6
6. $3/2 = 1.5 = 1\frac{1}{2}$
7. $6, -2$ (need both, either order)
8. -24
9. 168 or 168°

Varsity Meet 2 – November 20, 2013
FULL SOLUTIONS

ROUND 1

1. Let x be the proportion. Then

$$\begin{aligned} 350(1+x) &= 650(1-x) \\ 1000x &= 300 \\ x &= 0.3 \end{aligned}$$

The problem asks for a percentage, so $0.3 = \boxed{30\%}$.

2. Note that $4/9 = 0.\overline{4} \equiv 0.4\overline{4}$. Therefore, $4/9 - 0.0\overline{4} = 0.4$. Continuing in simplifying the complex fraction,

$$\begin{aligned} \frac{\frac{4}{9} - 0.0\overline{4} + \frac{1}{5}}{3} \div \frac{2}{7} &= \frac{0.4 + \frac{1}{5}}{3} \div \frac{2}{7} \\ &= \frac{0.6}{3} \div \frac{2}{7} \\ &= 0.2 \times \frac{7}{2} \\ &= \boxed{0.7}. \end{aligned}$$

3. Suppose the truck started with x packages. Then, make a table:

Stop No.	on truck	delivered
START	x	—
1	$(2/3)x$	$(1/3)x$
2	$(1/2)x$	$(1/6)x$
3	$(1/4)x$	$(1/4)x$

If 6 more packages are delivered at stop 3 than at stop 2, then $\frac{1}{4}x = \frac{1}{6}x + 6$, and $x = 72$. There are $x/4$ packages left on the truck after stop 3, so this is $72/4 = \boxed{18}$ packages.

ROUND 2

1. Since $5^{18} \times 2^{20} = 2^2 \times 10^{18}$, this is a $\boxed{4}$ followed by 18 zeros.

2. Rewrite $\frac{X+Y}{X}$ as $1+\frac{Y}{X}$. To maximize this quantity, choose the largest possible Y and the smallest possible X. This gives $1+\frac{80}{20}=1+4=\boxed{5}$.

3. In problems involving square roots, make sure to test for extraneous solutions.

$$
\begin{aligned}
\sqrt{4-\sqrt{x+7}} &= \sqrt{9-x} \\
4-\sqrt{x+7} &= 9-x \\
x-5 &= \sqrt{x+7} \\
x^2-10x+25 &= x+7 \\
x^2-11x+18 &= 0 \\
(x-9)(x-2) &= 0
\end{aligned}
$$

Possible roots are 9 and 2. Plugging back into the original equation, only $x=\boxed{9}$ works.

ROUND 3

1. We are given that $m\angle EAC=40°$ and $m\angle ABC=60°$. Since $\overline{DE}\parallel\overline{BC}$, $m\angle DAB=m\angle ABC=60°$. This leaves $m\angle CAB=180°-60°-40°=\boxed{80°}$.

2. The diagonals of a rhombus are perpendicular to each other. Connecting the midpoints of the sides gives a rectangle with side lengths 10 and 6 (use similar triangles to show that the side lengths of the rectangles are half the lengths of the corresponding parallel diagonal). Therefore, the diagonal of the rectangle is $\sqrt{10^2+6^2}=\sqrt{136}=\boxed{2\sqrt{34}}$.

3. Since trapezoid $ABCD$ has area 3, the height is 1. Let the distance between $\overline{AB}$ and $\overline{EF}$ be x. Then, proportionally, $EF=2+2x$.

   We are given that the areas of $ABFE$ and $EFCD$ are in the ratio 7 : 20, so the area of $ABFE$ is $\frac{7}{7+20}\cdot 3=\frac{7}{9}$. By the formula for the area of a trapezoid, it is also $\frac{1}{2}x\left(2+(2+2x)\right)=x(x+2)$. Therefore:

$$
\begin{aligned}
x(x+2) &= \frac{7}{9} \\
x^2+2x-\frac{7}{9} &= 0 \\
9x^2+18x-7 &= 0 \\
(3x+7)(3x-1) &= 0 \\
x &= \boxed{1/3}.
\end{aligned}
$$

ROUND 4

1. The common ratio is $-1/2$, so the zeroth term is $-192 = -3 \cdot 2^6$. Therefore, the 7^{th} term is $-3 \cdot 2^6 \cdot \left(-\frac{1}{2}\right)^7 = \boxed{3/2}$.

2. If $x+2$, $4x-2$, $6x-3$ forms an arithmetic sequence, then

$$\begin{aligned} (x+2)+(6x-3) &= 2(4x-2) \\ 7x-1 &= 8x-4 \\ 3 &= x \end{aligned}$$

so $A = 3$. Similarly, if $x+2$, $4x-2$, $6x-3$ forms a geometric sequence, then

$$\begin{aligned} (x+2)(6x+3) &= (4x-2)^2 \\ 6x^2+9x-6 &= 16x^2-16x+4 \\ 0 &= 10x^2-25x+10 \\ 0 &= 2x^2-5x+2 \\ 0 &= (2x-1)(x-2) \end{aligned}$$

so the possible values of G are $1/2$ and 2. However, plugging in $G = 1/2$ leads to an invalid geometric sequence (common ratio $r = 0$), so $G = 2$.

The problem asks for the value of $A + G$, so this is $3 + 2 = \boxed{5}$.

3. Rewrite using sigma notation and expand:

$$\begin{aligned} 1(99)+3(97)+5(95)+\ldots+99(1) &= \sum_{n=0}^{49}(2n+1)(99-2n) \\ &= \sum_{n=0}^{49}[-4n^2+196n+99] \\ &= -4\sum_{n=0}^{49}n^2+196\sum_{n=0}^{49}n+50\cdot 99. \end{aligned}$$

Now use the formula for the sum of squares

$$\sum_{n=0}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}$$

and the sum of an arithmetic series

$$\sum_{n=0}^{k} n = \frac{k(k+1)}{2}$$

to get

$$\begin{aligned}
-4\sum_{n=0}^{49} n^2 + 196\sum_{n=0}^{49} n + 50\cdot 99 &= -4\cdot\frac{49\cdot 50\cdot 99}{6} + 196\cdot\frac{49\cdot 50}{2} + 50\cdot 99 \\
&= 50\Big[-2\cdot 49\cdot 33 + 98\cdot 49 + 99\Big] \\
&= 50\Big[32\cdot 49 + 99\Big] \\
&= 50\Big[(1600-32)+(100-1)\Big] \\
&= 50\cdot 1667 \\
&= \boxed{83350}.
\end{aligned}$$

ROUND 5

1. From the rules of matrix multiplication, if the product **M** × **N** is defined, the number of columns in **M** must equal the number of rows in **N**. The product will have the same number of rows as **M** and the same number of columns as **N**.

   Therefore, in our problem, **B** must have 7 rows and 8 columns, a $\boxed{7 \times 8}$ matrix.

2. By the second equation, $x^2 = y + 7$. Substitute that into the first equation to obtain an equation only in y:

$$\begin{aligned}
y^2 + 7y + 12 &= 0 \\
(y+3)(y+4) &= 0 \\
y &= -3, -4
\end{aligned}$$

   Now use the original second equation to find that $x^2 = 4, 3$. Therefore, the four possible values of x are ± 2 and $\pm\sqrt{3}$. The problem asks for the product of these, which is $\boxed{12}$.

3. We will need to invert A and B, so first find the determinants: $\det A = -1$ and $\det B = 1$. Therefore, $A^{-1} = \begin{bmatrix} -3 & 8 \\ 2 & -5 \end{bmatrix}$ so $(A^{-1})^\top = \begin{bmatrix} -3 & 2 \\ 8 & -5 \end{bmatrix}$. Since $(A^\top)^{-1} = (A^{-1})^\top$ (proof[1]), $(A^\top)^{-1} + (A^{-1})^\top = \begin{bmatrix} -6 & 4 \\ 16 & -10 \end{bmatrix}$.

Now, $B^{-1} = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$, so $(B + B^{-1}) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, equal to its own transpose.

Add these up to find that $(A^\top)^{-1} + (A^{-1})^\top + (B + B^{-1})^\top = \boxed{\begin{bmatrix} 4 & 4 \\ 16 & 0 \end{bmatrix}}$.

TEAM ROUND

1. **METHOD I:** Let the two-digit number be $\underline{a}\,\underline{b} = 10a + b$, with $b = a + 1$, as required. The value of this number is $10a + (a + 1) = 11a + 1$. This number, with its digits reversed, is $10(a + 1) + a = 11a + 10$.

Therefore, we have that

$$\begin{aligned} (11a + 1)(11a + 10) &= 736 \\ 121a^2 + 121a + 10 &= 736 \\ 121a^2 + 121a - 726 &= 0 \\ a^2 + a - 6 &= 0 \\ (a + 3)(a - 2) &= 0 \\ a &= 2. \end{aligned}$$

This gives the original number as $\boxed{23}$.

METHOD II: The number with its digits reversed is only 9 greater than the number itself, so the two numbers are similar in magnitude. Estimate this magnitude by taking $\sqrt{736} \approx 27$, revealing that the two numbers must be 23 and 32 (make sure to check). The problem asks for the original number, with the tens digit one less than the units digit. This number is $\boxed{23}$.

2. Since 150% of the customers generate 125% of the revenue, the new price must be

$$\frac{125}{150} \cdot \$15 = \frac{5}{6} \cdot \$15 = \boxed{\$12.50}.$$

[1] Using the fact that $A^\top B^\top = (BA)^\top$, we have that $(A^\top)(A^{-1})^\top = (A^{-1}A)^\top = I^\top = I$. Therefore, $A^\top$ and $(A^{-1})^\top$ must be inverses of each other, so $(A^\top)^{-1} = (A^{-1})^\top$, as desired.

3. Borrowing the PRINCIPLE OF INCLUSION-EXCLUSION from Set Theory, we have that the answer is

$$999 - \left\lfloor \frac{999}{5} \right\rfloor - \left\lfloor \frac{999}{7} \right\rfloor + \left\lfloor \frac{999}{35} \right\rfloor = 999 - 199 - 142 + 28 = \boxed{686}.$$

($\lfloor x \rfloor$ is the greatest integer function, also called the floor function.)

4. **METHOD I:** Keep track of the hunters. The first hunter will hit 1 duck. The second hunter has a 3/4 chance of hitting a duck not already hit.

The third hunter has a 1/4 chance of hitting 3/4 of a duck (that is, the second hunter shot the same duck as the first hunter) and a 3/4 chance of hitting 2/4 of a duck (if the second hunter shot a different duck than the first hunter's). Therefore, the third hunter is worth $(1/4)(3/4) + (3/4)(2/4) = (3/4)^2$ of a duck. This is logical because in probability, it is not necessary to go through casework. The fourth hunter is worth $(3/4)^3$ of a duck.

Therefore, the expected number of ducks shot is[2]

$$1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 = \frac{175}{64}$$

and the expected number of survivors is $4 - \dfrac{175}{64} = \boxed{\dfrac{81}{64}}$.

METHOD II: Keep track of the ducks. In order for a duck to survive, it needs all of the hunters to target some other duck. There is a 3/4 chance that one hunter will target a different duck, so with 4 hunters, the chance of survival for a duck is $(3/4)^4$.

Every duck is equivalent, so the expected number of survivors is four times this:

$$4 \cdot \left(\frac{3}{4}\right)^4 = 4 \cdot \frac{81}{256} = \boxed{\frac{81}{64}}.$$

[This problem is often called the MONTANA DUCK HUNTER PROBLEM and is typically posed with 10 ducks and 10 hunters. As the number of ducks and hunters approaches infinity (with the same number of ducks as hunters), the chance of survival for a single duck approaches $1/e$, where $e = 2.71828\ldots$. Can you prove it?]

5. Solve the equation

$$\frac{x+3}{2} = \frac{3x}{4}$$

to get $x = \boxed{6}$.

[2] This can be summed up as a geometric series.

6. The key to this problem is to split the summation into two infinite geometric series:

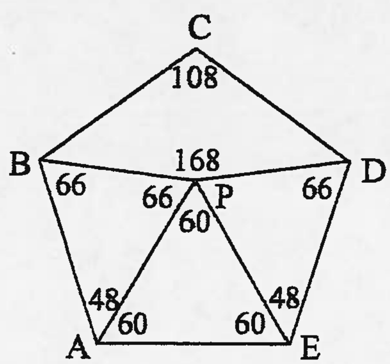
$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2^n-1}{3^n} &= \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n-\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\\ &= \frac{1}{1-2/3}-\frac{1}{1-1/3}\\ &= 3-3/2\\ &= \boxed{3/2}.\end{aligned}$$

7. This problem can be solved using VIETA'S FORMULAS. The information from Algernon tells us that the sum of the roots is $7+(-3)=4$, and the information from Jack tells us that the product of the roots is $4\cdot -3=-12$. The pair of numbers that satisfies both conditions is $\boxed{6,-2}$.

8. Use standard matrix operations to transform the first matrix into the second, and keep track of the effects on the determinant.
   - Take the transpose. No effect on the determinant.
   - Multiply the first column through by 2. The determinant is also multiplied by 2.
   - Interchange columns 2 and 3. The determinant changes sign.
   - Subtract 2 times column 2 from column 3. This standard row operation does not change the determinant.

   Therefore, the determinant of the second matrix is $12\cdot 2\cdot -1=\boxed{-24}$.

9. The interior angles of a regular pentagon measure 108°. Since $\triangle APE$ is equilateral, $AP=AE$. Since $ABCDE$ is a regular pentagon, $AE=AB$. Therefore, by transitivity, $AP=AB$, so $\triangle ABP$ is isosceles.



This gives $m\angle APB=m\angle EPD=66°$, so $m\angle BPD=360°-(66°+60°+66°)=\boxed{168°}$.